5) Hyperbolic surfaces, Tori and Lattès maps

Dynamics on hyperbolic surfaces.

We focus now on the study of the dynamics of holomorphic maps on hyperbolic Riemann surfaces.

Prop: For any map $f: X \to X$ of a hyperbolic surface, $\mathcal{J}(f) = \emptyset$.
In particular: f has no repelling nor parabolic periodic points, and basins of attraction have no boundary (hence if there is a basin of attraction $\mathcal{A}$, then $\mathcal{A} = X$).

Proof: Directly follows from the normality of $Hol(X, X)$ and the results seen in the previous section. □

We give here a more precise statement on the dynamics we can have.

Theorem. For any holomorphic map $f: X \to X$ on a hyperbolic surface, one of the following situations hold

- Attracting case: f has a contracting fixed point, then $X = \mathcal{A}$ its basin of attraction.

- Escape: If there exists an orbit without accumulation points, then the sequence (f^n) diverges uniformly from X.

- Finite order: if f has two distinct periodic points, then f has finite order: $\exists n \in \mathbb{N}$ so that $f^n = id$.

- Irrational rotation: Otherwise, (X, f) is a rotation domain: $X \simeq \mathbb{D}$, $\mathbb{D} \setminus \{0\} = \mathbb{D}^*$, or an annulus $\mathbb{A}_R = \{z \mid 1 < |z| < R\}$, and f is conjugated to an irrational rotation $z \mapsto e^{2\pi i \alpha} z$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

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Proof: Suppose we are in the hypothesis of Case 2, and there exists p_0 so that $O_f(p_0)$ has no accumulation point.

This is equivalent to the fact that p_0 diverges from X under f.

$\forall z\ \exists U_z$ s.t. $\# O_f(p_0) \cap U_z < \infty \iff \forall K$ compact s.t. $\# K \cap O_f(p_0) < +\infty$
$\exists N_z,\ f^n(p_0) \notin U_z\ \forall n \geq N_z$. $\quad (\exists N_K,\ f^n(p_0) \notin K\ \forall n \geq N_K)$

$\Rightarrow$ Cover K by U_z, $z \in K$, extract a finite covering (U_{z_j}), $N_K = \max_{z_j} n_{z_j}$.

$\Leftarrow$ $\forall z$, $\exists K$ compact s.t. $K \supset U_z \ni z$, and $N_z = N_K$.
some open

Hence the Poincaré distance ρ_X satisfies:

$$\lim_{n\to\infty} \rho_X(p_n, p_0) = +\infty \qquad (p_n = f^n(p_0))$$

Then for any $q_0 \in \overline{B_{\rho_X}(p_0, r)}$ ← closed ball (compact), we have that

$\rho_X(q_n, p_n) \leq r$ (ρ_X decreases), and $\rho_X(q_n, p_0) \geq \rho_X(p_n, p_0) - r \to +\infty$.

Hence f^n diverges uniformly from X.

Suppose now we are not in case 2, and $\rho_X(p_n, p_0) \not\to \infty$ ($\forall p_0$, but we just need one). Then there exists a compact subset $K \subset X$ and a subsequence $(f^{n_j}(p_0))_j$ contained in K. Up to extracting a subsequence, we may assume that $p_{n_j} \to p_\infty \in K$. $(j \in \mathbb{N}^*)$

Consider the sequence $g_j = f^{n_{j+1} - n_j}$ $(n_0 = 0)$, so that $f^{n_j} = g_{j-1} \circ \dots \circ g_0$

Set $r_j = \rho_X(p_{n_j}, p_\infty) \xrightarrow{j\to\infty} 0$

Since $g_j(p_{n_j}) = p_{n_{j+1}}$, $\rho_X(g_j(p_\infty), p_\infty) \leq \rho_X(g_j(p_\infty), \underset{g_j(p_{n_j})}{p_{n_{j+1}}}) + \rho(p_{n_{j+1}}, p_\infty)$

$\leq \rho_X(p_\infty, p_j) + \rho(p_{n_{j+1}}, p_\infty) = r_j + r_{j+1}$.

If $r = \max\{r_j, j \in \mathbb{N}^*\}$ (exists because $r_j \to 0$), then

$g_j(p_\infty) \in \overline{B_{\rho_X}(p_\infty, 2r)}$ a compact

Functions satisfying $\{g(\text{compact}) \subset \text{Compact}\}$ is a compact subset of $\mathrm{Hol}(X, X)$

given

$\Rightarrow$ Up to extracting a subsequence j_k, we may assume that $g_{j_k} \to g$ uniformly on compacts. Moreover, since $r_j + r_{j+1} \to 0$, we get that

$g(p_\infty) = p_\infty$

There are two cases depending on the classification of holomorphic maps between Riemann surfaces:

- Distance decreasing case: $\rho_X(f(p), f(q)) < \rho_X(p, q)$ $\forall p \neq q$.

Then $\rho_X(g_j(p), g_j(q)) < \rho_X(p, q)$ (being g_j an iterate of f)

and being the limit $g_{j_k} \to g$ uniform on compact subsets, we have that

$\rho_X(g(p), g(q)) < \rho_X(p, q)$.

Since f and g commute, f maps $p_\infty = g(p_\infty)$ to $f(p_\infty) =$

$= f(g(p_\infty)) = g(f(p_\infty))$.

g has a unique fixed point p_∞ (if it has 2, it cannot decrease distance)

Hence $f(p_\infty) = p_\infty$.

Since f decreases distances on compact subsets ($B_{\rho_X}(p_\infty, r)$), f is a contraction, and p_∞ is a contracting fixed point, with attracting basin A.

Since attracting basins have no boundary, $A = X$, and we are in case 1.

- Distance preserving case (isometries)

We assume that f is a local isometry (f is either an automorphism or a covering map)

As before, the limit map g also satisfies

$\rho_X(g(p), g(q)) = \rho_X(p,q)$ for $\rho_X(p,q) << 1$.

g has a fixed point p_∞, whose multiplier must be $g'(p_\infty) = e^{2\pi i \alpha}$ $\alpha \in \mathbb{R}$.

(cannot be repelling, nor contracting, as it is not a local isometry).

Notice that g^n has multiplier $e^{2\pi i m \alpha}$. We may pick m so that

$e^{2\pi i m \alpha} \sim 1$ So there exists a subsequence g^{m_j} so that $e^{2\pi i m_j \alpha} \to 1$.

By normality of this family (X hyperbolic), we may assume up to subsequences that $g^{m_j} \to g_\infty$, with $g_\infty'(p_\infty) = \lim {g^{m_j}}'(p_\infty) = 1$.

Let $pr : \mathbb{D} \to X$ be the universal covering, and $G : \mathbb{D} \to \mathbb{D}$ the lift of g_∞ so that $G(p) = p$ for some $p \in pr^{-1}(p_\infty)$.

By Schwarz Lemma, we get that $G \equiv id_{\mathbb{D}}$, hence $g_\infty \equiv id_X$.

~~Notice~~ To sum up: we have: a sequence $f^{n_j} \to g$, and $g^{m_k} \to g_\infty = id$.

(the first is up to considering a subsequence so that the n_j are increasing)

Then $f^{n_j \cdot m_k} \to g^{m_k} \to g_\infty = id_X$

Hence we may assume we have a sequence of iterates $f^{n_j} \to id$ on S. (if local isometry)

lemma
[We want to prove that then, either f has finite order (case 3), or
$X \sim \mathbb{D}, \mathbb{D}\setminus\{0\}, A_r$, and f is conjugated to an irrational rotation.]

Proof of this point:

Step 1: f must be a conformal automorphism.

In fact, f is injective: if $\exists p \neq q$, $f(p) = f(q)$, then

$f^{n_j}(p) = f^{n_j}(q) \Rightarrow p = q$. absurd ($f^{n_j} \to id$)

f is surjective: Suppose $\exists p \in X \setminus f(X)$.

Let B be a closed disc centred at p of positive radius.

Any map g sufficiently close to id must map B close to itself and hence it contains p in its image.

Hence $f^{n_j}(X) \ni p$ for $k \gg 0$, a contradiction.

Step 2: ~~simply~~ connected case.

$f: \mathbb{D} \to \mathbb{D}$ an automorphism ~~with~~ ~~a fixed~~ not in case 2 $\Rightarrow$ it must be an elliptic automorphism (~~elliptic case~~) $\leadsto$ f is conjugated to ~~a~~ ~~[illegible]~~ rotation.

If the rotation is rational, then we are in case 3, if not, we are in case 4 and $X \simeq \mathbb{D}$.

Step 3: X is not simply connected. We show that $\Gamma = \pi_1(X)$ is abelian.

Let $pr: \tilde{X} \to X$ be the universal covering, $\tilde{X} = \mathbb{D}$.

We lift f to maps $F: \mathbb{D} \to \mathbb{D}$ ... so that f^{n_j} lifts to F^{n_j}.

Since $f^{n_j} \to id$, then $F^{n_j} \to id \mod (\Gamma)$, $\forall k \in \mathbb{N}$

there exists $\gamma_j \in \Gamma$ so that $F_j := \gamma_j \circ F^{n_j} \to id$. uniformly on compact sets of $\mathbb{D}$.

Each F_j induces a group homomorphism $\Phi_j: \Gamma \to \Gamma$, by the property $F_j \circ \gamma = \Phi_j(\gamma) \circ F_j$ (F_j and $F_j \circ \gamma$ cover f^{n_j}, hence they differ by a deck transformation). This can be done fiber by fiber, and being fibers discrete, it implies that $\Phi_j(\gamma)(p)$ does not depend on p)

In particular $\Phi_j(\gamma) = F_j \circ \gamma \circ F_j^{-1}$. For $j \gg 1$, F_j is close to the identity, and, being Γ discrete, we must have $\Phi_j(\gamma) = \gamma$, i.e. $\Phi_j = id$ $\forall j \gg 0$

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If for some j, $F_j = \mathrm{Id}$, then $f^{n_j} = \mathrm{Id}$, and we are in case 3.

Suppose that this is not the case, and $F_j \neq \mathrm{Id}$ $\forall j$

Lemma $f, g \in \mathrm{Aut}(\mathbb{D}) \setminus \{\mathrm{Id}\}$ commute $\iff \mathrm{Fix}(f|_{\overline{\mathbb{D}}}) = \mathrm{Fix}(g|_{\overline{\mathbb{D}}})$.

Proof: Recall that $f \in \mathrm{Aut}(\mathbb{D}) \setminus \{\mathrm{Id}\}$ is either

- elliptic: $\mathrm{Fix}(f|_{\overline{\mathbb{D}}}) = \{p\}$ $p \in \mathbb{D}$
- parabolic " $= \{p\}$ $p \in \partial\mathbb{D}$
- hyperbolic " $= \{p_1, p_2\}$, $p_j \in \partial\mathbb{D}$

$\Rightarrow$ If $p = f(p)$, then $g(p) = g(f(p)) = f(g(p)) \Rightarrow g(p) \in \mathrm{Fix}(f)$

Hence $g(\mathrm{Fix}(f)) \subseteq \mathrm{Fix}(f)$, and being g an automorphism and $\# \mathrm{Fix}(f) < \infty$, we get $g(\mathrm{Fix}(f)) = \mathrm{Fix}(f)$.

If $\mathrm{Fix}(f) = \{p\}$, then $g(p) = p$ and $\mathrm{Fix}(f) \subseteq \mathrm{Fix}(g)$

If f is elliptic so g also is, and reversing the role of f and g, we get $\mathrm{Fix}(f) = \mathrm{Fix}(g)$.

If f is parabolic, either g is parabolic and again $\mathrm{Fix}(f) = \mathrm{Fix}(g)$, or g is hyperbolic. In this case, let $p_2 \in \mathrm{Fix}(g) \setminus \mathrm{Fix}(f)$. Then $f(p_2) = f(g(p_2)) = g(f(p_2)) \Rightarrow f(p_2) = p_2$ (contradiction) or $f(p_2) = p_1$ contradiction.

If f is hyperbolic, set $\mathrm{Fix}(f) = \{p_1, p_2\}$. Then either:

- $g(p_1) = p_1$, $g(p_2) = p_2$, and hence $\mathrm{Fix}(f) = \mathrm{Fix}(g)$, or:
- $g(p_1) = p_2$, $g(p_2) = p_1$. But the only automorphism with periodic (not fixed) orbits in $\overline{\mathbb{D}}$ are elliptic, which gives a contradiction (it would imply f elliptic).

$\Leftarrow$ Suppose $\mathrm{Fix}(f) = \mathrm{Fix}(g)$

- elliptic case: Up to change of coordinates, we may assume $\mathrm{Fix}(f) = \{0\}$.

$\Rightarrow f(z) = \lambda z$; $g(z) = \mu z$. For some $\lambda, \mu \neq 0$, and $f \circ g = g \circ f$.

- parabolic case: we may ~~assume~~ work on $\mathbb{H}$, and assume that $Fix(f)=\{\infty\}$ 5.7

Then $f(z)=z+a$; $g(z)=z+b$ $a,b\in\mathbb{R}\setminus\{0\}$, and $f\circ g=g\circ f$

- hyperbolic case; again work on $\mathbb{H}$, and assume $Fix(f)=\{0;\infty\}$.

Then $f(z)=\lambda z$; $g(z)=\mu z$, $\lambda,\mu\in\mathbb{R}_+^*\setminus\{1\}$, and $f\circ g=g\circ f$ $\square$

End of proof of main thm: We have that $F_j\circ\gamma=\gamma\circ F_j$ $\forall\gamma\in\Gamma$, $\forall j>>0$

Take any $\gamma\neq id$. Then by the lemma, $\exists j_0$ st $\forall j\geq j_0$, $Fix(\gamma)=Fix(F_j)=A$.

Then $\forall\gamma'\neq\gamma, id$, we have $Fix(\gamma')=\underset{j>>0}{Fix}(F_j)=A$.

Hence $\forall\gamma,\gamma'$, $\gamma\circ\gamma'=\gamma'\circ\gamma$, and Γ is abelian.

Step 4: Conclusion (*)

Recall that Γ cannot have elliptic elements (they don't act freely).

If Γ contains parabolic elements, then all are, sharing the same fixed point

$A=\{p\}$. We may work on $\mathbb{H}$ and assume $p=\infty$.

$\gamma\in\Gamma$ is of the form $z\mapsto z+a_\gamma$ $a_\gamma\in\mathbb{R}\setminus\{0\}$.

The set $\{a_\gamma\}$ must be discrete $\Rightarrow$ it is generated by a single $a\in\mathbb{R}\setminus\{0\}$.

Hence $\Gamma=\langle z\mapsto z+a\rangle$. Up to linear change of coordinates $z\mapsto\dfrac{az}{2\pi}$,

we may assume $a=2\pi$.

(*) Since F_j commutes with γ $\forall\gamma\in\Gamma$, ~~and~~ $\overset{F_j \text{ is a translation}}{=}$ $F_j=\gamma_j\circ F^{n_j}$ $\Rightarrow$ F^{n_j} is itself

a translation. Since F commutes with F^{n_j}, F is a translation

too, $F(z)=z+\alpha$.

(*) Then we have: $\Gamma=\langle z\mapsto z+2\pi\rangle$, $F(z)=z+\alpha$.

The universal covering here is given by $p: \mathbb{H}\longrightarrow\mathbb{D}^*$
$z\mapsto e^{iz}$

F corresponds to $f(w)=e^{i\alpha}w$, which is a rotation.

Assume now that Γ has hyperbolic elements.

As before, $\Gamma \setminus \{id\}$ is made by hyperbolic elements, $A = \{0, \infty\}$ on suitable coordinates, $\gamma(z) = \mu_\gamma z$, $\mu_\gamma \in \mathbb{R}_+^* \setminus \{1\}$.

As before, $\Gamma = \langle z \mapsto \mu z \rangle$, for some μ, that we may suppose > 1.

Again, from F_j commuting with Γ we infer F commuting with $\forall \gamma \in \Gamma$ and $F(z) = \lambda z$ for some $\lambda \in \mathbb{R}_+ \setminus \{1\}$.

Notice that $\mathbb{H}/\Gamma$ is isomorphic to $\mathbb{H}'/\Gamma'$, with $\mathbb{H}' = \mathbb{R} \times i(0, \pi)$, and

$\Gamma' = \langle z \mapsto z + \ln \mu \rangle$, with the isomorphism induced by $\mathbb{H}' \to \mathbb{H}$, $z \mapsto e^z$.

In this case f corresponds to $z \overset{F'}{\mapsto} z + \ln \lambda$.

Then $pr' : \mathbb{H}' \to \mathbb{A}_r$, $z \mapsto e^{-iz\frac{2\pi}{\ln \mu}}$ is the universal covering, where $\mathbb{A}_r = \{1 < |z| < r\}$,

$$r = \frac{2\pi^2}{\ln \mu}.$$

In this case, F' corresponds to $f(w) = e^{-i2\pi \frac{\ln \lambda}{\ln \mu}} w$, which is again a rotation

$\square$

Later, we will apply this classification of dynamics on hyperbolic Riemann surfaces to f-invariant Fatou components U for $f : \hat{\mathbb{C}} \circlearrowleft$, $\deg f \geq 2$.

In fact, being $J(f) \neq \emptyset$ and infinite, any f-invariant Fatou component is Hyperbolic.

With respect to the classification, Case 1 will happen for every contracting fixed point and its immediate basin of attraction.

We will show examples where orbits escape (toward a parabolic fixed point), while case 3 cannot happen, since it would imply f to have finite order, against the hypothesis $\deg f \geq 2$.

For case 4, notice that U cannot be $\simeq \mathbb{D}^*$, since it would imply the existence of an isolated point in $J(f)$.

If $\Phi : \mathbb{D}^* \to U \subset \hat{\mathbb{C}}$ is the isomorphism, $0 \in \mathbb{D}$ is an isolated singularity, and it cannot be essential (against injectivity), hence it extends to a map $\Phi : \mathbb{D} \to \hat{\mathbb{C}}$.

Def: Let $f : \hat{\mathbb{C}} \circlearrowleft$ be a rational map of degree $d \geq 2$.

A Siegel disk is a connected component $U \subset F(f)$ where (U, f) is isomorphic to $(\mathbb{D}, z \mapsto \lambda z)$, $|\lambda| = 1$ (λ not a root of 1).

An Herman ring is a connected component $U \subset F(f)$ with $(U, f) \simeq (\mathbb{A}_R, z \mapsto \lambda z)$ $|\lambda| = 1$ λ not a root of unity, $\mathbb{A}_R$ annulus.

We will see later the existence of rational maps exhibiting Siegel disks / Herman rings.

The latter cannot appear for polynomial mappings.

Proposition: Let $P \in \mathbb{C}[z]$ be a polynomial of degree $d \geq 2$. Then P has no Herman rings.

Proof: Let $A \subset F(f)$ be a Herman ring for $f(z) = P(z)$.

Since ∞ is a fixed point for f, $A \not\ni \infty$.

Then A has a outer boundary C^+, and a inner boundary C^-.

~~By~~ Let B be the bounded connected component of $\hat{\mathbb{C}} \setminus C^+$.

By maximum modulus principle, $|f^n|$ takes its $\max|_{\bar{B}}$ on C^+.

But by continuity $f(C^+) \underset{(=)}{\subseteq} C^+$, and $f^n|_{\bar{B}}$ is bounded.

This is a contradiction, since $C^- \subset J(f)$ and $J(f) = \partial \mathcal{A}_\infty(f)$

(the basin of attraction of infinity). □

For maps $f : \mathbb{D} \to \mathbb{D}$ (or more generally $f : U \to U$, U hyperbolic simply connected, $U \subset \hat{\mathbb{C}}$), one can say more about the escape case.

Theorem (Denjoy-Wolff)

Let $f : \mathbb{D} \to \mathbb{D}$ be any holomorphic map. Then either:

(a) f is a "rotation" (i.e., conjugated to a rotation at 0) around some $z_0 \in \mathbb{D}$,

or (b) f^n converges uniformly on compact subsets to a constant function $f^{\infty}_{\infty}(z) = c_0$, with $c_0 \in \overline{\mathbb{D}}$ (if $c_0 \in \partial\mathbb{D}$, f^n escape towards c_0 ...)

Proof. $\forall \varepsilon > 0$ small enough consider $f_\varepsilon(z) = (1-\varepsilon) f(z)$,

$f(\mathbb{D}) \subset D(0, 1-\varepsilon) \subset \overline{D(0, 1-\varepsilon)}$.

Being $K_\varepsilon = \overline{D(0, 1-\varepsilon)}$ compact, and $f_\varepsilon|_{K_\varepsilon}$ a contraction, f_ε admits a unique fixed point $z_\varepsilon \in K_\varepsilon$. ($f_\varepsilon \xrightarrow[\varepsilon \to 0]{} f$ unif. on compacts)

Being $\overline{\mathbb{D}}$ compact, we may find $\varepsilon_j \to 0$ so that $z_{\varepsilon_j} \to z_\infty \in \overline{\mathbb{D}}$.

If $z_\infty \in \mathbb{D}$, it is a fixed point for f, and the result follows from the classification given above.

Assume $z_\infty \in \partial\mathbb{D}$. Pick any $p \in \mathbb{D}$, we want to show that $f^n(p) \to z_\infty$.

$\forall j$, set $r_j = \rho_{\mathbb{D}}(p, z_{\varepsilon_j})$ and $B_j = \overline{D_{\rho_{\mathbb{D}}}(z_{\varepsilon_j}, r_j)} \ni p$ (on the boundary)

Since f_{ε_j} reduces $\rho_{\mathbb{D}}$, $f_{\varepsilon_j}(B_j) \subset B_j$.

Claim: B_j are round discs in the euclidean topology. To see this: $D_{\rho_{\mathbb{D}}}(0, r)$ is a disc, we saw this.

If $\phi_{-a}(z) = \dfrac{z+a}{1+\bar{a}z}$, $\phi_{-a}(D_{\rho_k}(0,r)) = D_{\rho_k}(a,r)$.

~~If~~ $\phi_{-a}(\partial D_{\rho_k}(0,r))$ must be a circle, ~~[illegible]~~ because Moebius transformations send circles to circles or lines. The result follows.

Then $B_i \to B_\infty$ the disc tangent to z_0 and so that $p \in \partial B_\infty$. (this disc is called a Horodisk).

But we know that orbits escape, so we have $f^n(p) \to z_0$. $\square$

For arbitrary $U \subset \hat{\mathbb{C}}$, we have:

Prop: let $U \subset \hat{\mathbb{C}}$ be a open connected hyperbolic subset of $\hat{\mathbb{C}}$, and that $f: U \to U$ extends continuously to ∂U with at most finitely many fixed points in ∂U (like if $f: \hat{\mathbb{C}} \circlearrowleft$).

If some orbit of $f|_U$ has no accumulation point in U (escape case) then all orbits in U converge (in $\bar{U}$) to a fixed point in ∂U. The convergence is uniform on compact subsets.

Reformulation.

Prop: let $f: U \circlearrowleft$, U hyperbolic Riemann surface.

Assume we are in the escape case. If moreover:

- $U \subset X$, X compact Riemann surface, f extends continuously to $\bar{U} \subset X$
- $f|_{\partial U}$ has finitely many fixed points

Then $\exists p \in \partial U$, f^n converges locally uniformly on $\bar{U}$ to the constant $z \mapsto p$.

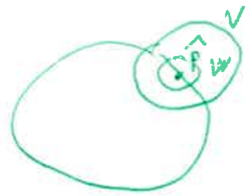
Proof: we are in the escape case, then a (all) orbits diverges from U.

(We would like that any limit point of a orbit must be fixed by f.
By using the fact that $Fix(f|_{\partial U})$ is finite ~~and~~ we would like to use continuity arguments to deduce the uniqueness.
To do so we cannot work with the orbit of a single point (think of "expansion o involution").)

Take a diverging orbit $(p_n)_{n\in\mathbb{N}}$. Construct a path p_t, $t\in[0,+\infty)$, by picking any path $p:[0,1]\to U$ joining p_0 and p_1, and defining $p(t+1)=f(p(t))$ recursively on $\lfloor t\rfloor$.

Notice that if $\delta = \mathrm{diam}_{\rho_U} p([0,1])$, then $\mathrm{diam}_{\rho_U} p([n,n+1]) \le \delta \ \forall n\in\mathbb{N}$.

Let $\hat{p}$ be any accumulation point of $p(t)$ in ∂U, as $t\to+\infty$.



$\forall V\ni \hat{p}$ open $\exists W\ni \hat{p}$, $W\subset V$, so that $\forall I\subset U$
$I\cap W\neq\emptyset$, $\mathrm{diam}_{\rho_U}(I)<\delta \Rightarrow I\subset V$.
($\mathrm{dist}_{\rho_U}\to\infty$ when we approach the boundary)

$\Rightarrow$ We have that $\forall N\ \exists n>N$ s.t. $p([n,n+1])\subset V$.

Since $f(p(n))=p(n+1)$, it follows by continuity that $f(\hat{p})=\hat{p}$.

Hence $\underbrace{Acc(p)}_{\text{connected}} \subset \underbrace{Fix(f|_{\partial U})}_{\text{finite}} \Rightarrow f^n(p_0)\to\hat{p}$.

$Acc(p) := \bigcap_{t>0} \overline{p([t,+\infty))}$ ...

$\forall$ other $q_0\in U$, let $r=\mathrm{dist}_{\rho_U}(p_0,q_0) \Rightarrow \mathrm{dist}_{\rho_U}(p_n,q_n)\le r$, and we get that $f^n(q_0)\to\hat{p}$ (always for the behavior of ρ_U at the boundary) □

It is easy to check that the convergence is uniform on compacts of U.